

# Experimental test of the time-dependent Wigner inequalities for neutral pseudoscalar meson systems

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## Abstract

Recently a new class of time-dependent Bell inequalities in Wigner form was introduced. The structure of the inequalities allows experimental studies of quantum and open quantum systems in external fields. In this paper we study the properties of the time-dependent Wigner inequalities using the time evolution of neutral pseudoscalar mesons. It is shown that it is always possible to find a range of parameters to test for violation in an experimentally accessible area. The effect of the relaxation of the inequalities for large time scales is demonstrated.

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## I. INTRODUCTION

In [1] a new class of time-dependent Bell inequalities in Wigner form was introduced. The structure of the inequalities allows experimental studies of the quantum and open quantum systems in an external fields. In this paper we study the properties of the time-dependent Wigner inequalities using the time evolution of neutral pseudoscalar mesons.

In [2] the following question was raised for the first time: can the properties of a macro-system which in quantum theory are described by noncommuting operators be simultaneously the elements of reality (i.e. to exist simultaneously), even if these properties can not be measured by any macro-device? According to the Copenhagen interpretation of quantum mechanics, the answer is no [3]. Bohr demonstrated the fundamental difference between the statistical and the Copenhagen interpretations, but did not give a conclusive proof [2]. An attempt to move the problem of the simultaneous existence of the elements of physical reality from the gedanken to the experimental realm has been made by J. Bell [4–6]. Bell’s idea has been further developed by Clauser, Horne, Shimony, and Holt [7]. Since then the idea has been thoroughly studied, developed, and criticized by many Bell’s proponents and opponents [8–11].

We address here the question of how to express the fact, that some set of a micro-system characteristics (observables) is simultaneously the set of elements of the physical reality, even if that set can not be simultaneously measured by any macro-device. One possibility is to assert that the joint probability of simultaneous existence of members of the set is non-negative. In quantum mechanics the probability is the universal object. Such a proposition (however without the clear statement) was used by Bell in [4]. The idea of the non-negativity of the joint probability was proposed by E. Wigner [12].

Following Wigner’s approach, let us suppose that a quantum system decays at the time  $t_0$  into two subsystems “1” and “2”, each having three observables  $a$ ,  $b$ , and  $c$ . Let each observable be dichotomic variable (able to have only two distinct values). For simplification let us set these values to  $\pm 1$ . We will use the following definitions: if the observable  $a$  is equal to  $+1$  we denote this as  $a_+^{(1)}$ , and so on. At the time  $t_0$  let all three observables to satisfy the anticorrelation condition

$$n_{\pm}^{(1)}(t_0) = -n_{\mp}^{(2)}(t_0), \quad (1)$$

where  $n^{(i)} = \{a^{(i)}, b^{(i)}, c^{(i)}\}$  and  $i = \{1, 2\}$ .

An example of such a system is a pseudoscalar particle which decays into a fermion-antifermion pair with a Hamiltonian

$$\mathcal{H}^{(PS)}(x) = g \varphi(x) (\bar{f}(x) \gamma^5 f(x))_N, \quad (2)$$

which automatically provides full anticorrelation of the fermions' spin projections onto any direction. In (2), the  $\varphi(x)$  is a pseudoscalar field, and  $\bar{f}(x)$  and  $f(x)$  are fermionic fields.

Let  $a$ ,  $b$ , and  $c$  exist simultaneously as the elements of physical reality, i.e. any of their double and triple joint probabilities are non-negative. Then, under the assumption of locality at time  $t_0$  and using Kolmogorov's axiomatics of probability theory, the following inequality can be obtained [1]:

$$w(a_+^{(2)}, b_+^{(1)}, t_0) \leq w(c_+^{(2)}, b_+^{(1)}, t_0) + w(a_+^{(2)}, c_+^{(1)}, t_0). \quad (3)$$

If one drops the  $t_0$  from (3), it transforms into the Wigner inequality [12]

$$w(a_+^{(2)}, b_+^{(1)}) \leq w(c_+^{(2)}, b_+^{(1)}) + w(a_+^{(2)}, c_+^{(1)}). \quad (4)$$

We name the inequalities (3) and (4) time-independent or static Wigner inequalities, to distinguish them from the time-dependent inequalities in [1]. The observables  $a$ ,  $b$ , and  $c$  may correspond to non-commuting operators and, hence, cannot be simultaneously measured by any macro-device. As was mentioned in [13], the Wigner inequalities are more suitable for experimental tests, due to the fact that the probabilities, unlike the correlators, are well defined in both non-relativistic quantum theory and in quantum field theory.

In [1], a new class of Wigner inequalities was obtained with a direct dependence on time:

$$\begin{aligned} & w(a_+^{(2)}, b_+^{(1)}, t) \leq \\ & \leq w(a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t)) \left( w(b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t)) + w(b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t)) \right) w(a_+^{(2)}, c_+^{(1)}, t_0) + \\ & + w(a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t)) \left( w(b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t)) + w(b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t)) \right) w(a_-^{(2)}, c_+^{(1)}, t_0) + \\ & + w(b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t)) \left( w(a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t)) + w(a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t)) \right) w(c_+^{(2)}, b_+^{(1)}, t_0) + \\ & + w(b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t)) \left( w(a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t)) + w(a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t)) \right) w(c_+^{(2)}, b_-^{(1)}, t_0). \end{aligned} \quad (5)$$

For closed quantum systems,  $w(a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t)) = w(b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t)) = 0$ , while  $w(a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t)) = w(b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t)) = 1$ . Hence (5) reduces to (3), as it should be

from physical point of view. The inequality (3), in turn, is equivalent to the time-independent inequality (4).

The Leggett-Garg inequalities are based on the idea of macroscopic realism [14]. They resemble Bell inequalities [7] but, instead of the simultaneous correlation of the two observables, they concern the correlation between the values of a single observable at different points of time. In experiment they are closely related to the weak (non-invasive) measurements [15, 16], involving, for example, nanomechanical resonators [17] or discrete lattices [18].

The main distinction between the Leggett-Garg inequalities and (5) is the fact that the test of (5) does not require weak measurements. The measurement is fully invasive. That opens the experimental possibility to verify (5) at contemporary high energy physics detectors like LHCb, ATLAS, CMS, and Belle II.

We study here the violation of the time-dependent inequality (5) in systems of neutral pseudoscalar mesons. Many papers attempt to include time-dependence into the static Wigner inequalities [4, 7, 12], and studies of the obtained time-dependent inequalities in quantum theory are available [19] – [30]. A number of authors [22] – [26] try to adapt Wigner inequalities for oscillations of pseudoscalar mesons, usually the neutral  $K$ -mesons. First, these adapted inequalities are studied in terms of “flavour”–“ $CP$ -violation”–“states with defined masses and lifetimes”. This idea was introduced in [22]. The time-dependence is included by substitution of probabilities calculated in the framework of quantum mechanics. In this case Wigner inequalities become inequalities among the parameters  $\varepsilon$  and  $\varepsilon'$  of  $CP$ -violation. The violation of these inequalities is small and is currently beyond experimental reach [22]. Secondly, there are attempts to include additional correlation functions which depend on time difference [26]. A third way, introduced in [19] – [21], is based on the requirements of the causality principle and locality; however the obtained inequalities are not general and are suitable only for the specific situation of the oscillations of neutral mesons. Finally, in [26] – [30], special versions of time-dependent inequalities in the form [7] are introduced, but there are certain difficulties with their violation in quantum mechanics.

To demonstrate the distinction between (5) and (4), we will apply them to the problem of oscillations of neutral pseudoscalar mesons  $M = \{K, D, B_q\}$ ,  $q = \{d, s\}$ . In this case the static inequalities (4) are either not violated at all, or the scale of the violation is beyond experimental reach [22]. The violation of the time-dependent inequality (5), on the other

hand, can be significantly enhanced by a proper choice of parameters and hence allows experimental tests.

## II. STATIC WIGNER INEQUALITIES FOR OSCILLATIONS OF NEUTRAL PSEUDOSCALAR MESONS

The key idea of the static Bell inequalities for the task at hand was suggested in [19, 22, 23] and developed in [25, 26, 28].

The essence of the idea is that there are three naturally provided “directions” whose projection operators do not commute. The first is the flavour of a pseudoscalar meson. For example, for the  $B_q$ -mesons, we consider the projections onto the states  $|B_q\rangle = |\bar{b}q\rangle$  and  $|\bar{B}_q\rangle = |b\bar{q}\rangle$ . We define the operators for charge ( $\hat{C}$ ) and spatial ( $\hat{P}$ ) conjugation onto the states in the flavour space as

$$\hat{C}\hat{P}|M\rangle = e^{i\alpha}|\bar{M}\rangle \quad \text{and} \quad \hat{C}\hat{P}|\bar{M}\rangle = e^{-i\alpha}|M\rangle,$$

where the  $\alpha$  is a non-physical arbitrary real phase of  $CP$ -violation. This phase should be excluded from any experimentally testable inequalities.

The second “direction” is the states with defined values of  $CP$ -parity, i.e. the states

$$|M_1\rangle = \frac{1}{\sqrt{2}} (|M\rangle + e^{i\alpha}|\bar{M}\rangle), \quad |M_2\rangle = \frac{1}{\sqrt{2}} (|M\rangle - e^{i\alpha}|\bar{M}\rangle),$$

which have positive and negative  $CP$ -parity accordingly.

The third “direction” is defined by the states with fixed values of mass and lifetime

$$|M_L\rangle = p \left( |M\rangle + e^{i\alpha} \frac{q}{p} |\bar{M}\rangle \right) \quad \text{and} \quad |M_H\rangle = p \left( |M\rangle - e^{i\alpha} \frac{q}{p} |\bar{M}\rangle \right).$$

The latter two states are the proper vectors of the non-hermitian hamiltonian (for which  $CPT$ -symmetry is preserved):

$$\hat{H} = \begin{pmatrix} \mathcal{H} & H_{12} e^{-i\alpha} \\ H_{21} e^{i\alpha} & \mathcal{H} \end{pmatrix} = \begin{pmatrix} m - i/2 \Gamma & (m_{12} - i/2 \Gamma_{12}) e^{-i\alpha} \\ (m_{12}^* - i/2 \Gamma_{12}^*) e^{i\alpha} & m - i/2 \Gamma \end{pmatrix},$$

with the proper values

$$\begin{aligned} E_L &= m_L - i/2 \Gamma_L = \mathcal{H} - \sqrt{H_{12} H_{21}} = \mathcal{H} + q/p H_{12} \quad \text{and} \\ E_H &= m_H - i/2 \Gamma_H = \mathcal{H} + \sqrt{H_{12} H_{21}} = \mathcal{H} - q/p H_{12} \end{aligned}$$

accordingly (here and subsequently we use the natural system of units in which  $\hbar = c = 1$ ). The states  $|M_L\rangle$  and  $|M_H\rangle$  are not orthogonal to each other. The complex coefficients  $p$  and  $q$  are subjected to the standard normalization condition:

$$\langle M_L | M_L \rangle = \langle M_H | M_H \rangle = |p|^2 + |q|^2 = 1. \quad (6)$$

We define

$$\begin{aligned} \Delta M &= M_H - M_L = -2 \operatorname{Re} \left( \frac{q}{p} H_{12} \right), \\ \Delta \Gamma &= \Gamma_H - \Gamma_L = 4 \operatorname{Im} \left( \frac{q}{p} H_{12} \right). \end{aligned}$$

Note that the definition of  $\Delta \Gamma$  in the current work is oppositely signed relative to the definition in [31].

To automatically satisfy (6), we introduce a new variable  $\beta$ , for which

$$|p| = \cos \beta; \quad |q| = \sin \beta; \quad \text{and} \quad \frac{q}{p} = \operatorname{tg} \beta e^{i\zeta} \equiv r e^{i\zeta}, \quad \beta \in [0, \pi/2].$$

Then

$$|M_L\rangle = p (|M\rangle + e^{i(\alpha+\zeta)} \operatorname{tg} \beta |\bar{M}\rangle) \quad \text{and} \quad |M_H\rangle = p (|M\rangle - e^{i(\alpha+\zeta)} \operatorname{tg} \beta |\bar{M}\rangle).$$

Decay of a neutral vector state  $1^{--}$  into an  $M\bar{M}$ -pair (e.g.  $\phi(1020) \rightarrow K\bar{K}$  or  $\Upsilon(4S) \rightarrow B\bar{B}$ ) defines a flavour-entangled wave function of the  $M\bar{M}$ -system at  $t = t_0$ :

$$|\Psi(t_0)\rangle = \frac{1}{\sqrt{2}} \left( |M\rangle^{(2)} |\bar{M}\rangle^{(1)} - |\bar{M}\rangle^{(2)} |M\rangle^{(1)} \right). \quad (7)$$

The distinction between the first and the second meson can be provided by their direction in the experimental device, see for example [22, 25, 26].

We obtain the static Wigner inequalities following the logic of [26]. We make the correspondence  $a_+ \rightarrow M_1$ ,  $a_- \rightarrow M_2$ ,  $b_+ \rightarrow \bar{M}$ ,  $b_- \rightarrow M$ ,  $c_+ \rightarrow M_H$  and  $c_- \rightarrow M_L$ . Then (4) becomes:

$$w(M_1^{(2)}, \bar{M}^{(1)}, t_0) \leq w(M_1^{(2)}, M_H^{(1)}, t_0) + w(M_H^{(2)}, \bar{M}^{(1)}, t_0). \quad (8)$$

Substitution of probabilities (A1) from Appendix A into (8) leads to:

$$|q|^2 - |p|^2 \leq |p + q|^2. \quad (9)$$

As there is no unambiguous correspondence between the projections of the meson states onto various “directions” and the projections from (4), we set  $b_+ \rightarrow M$  and  $b_- \rightarrow \bar{M}$  while keeping the  $a_\pm$  and  $c_\pm$ . Then (4) becomes:

$$w(M_1^{(2)}, M^{(1)}, t_0) \leq w(M_1^{(2)}, M_H^{(1)}, t_0) + w(M_H^{(2)}, M^{(1)}, t_0), \quad (10)$$

and, taking into account (A1):

$$|p|^2 - |q|^2 \leq |p + q|^2. \quad (11)$$

One can merge (9) and (11) as:

$$||p|^2 - |q|^2| \leq |p + q|^2. \quad (12)$$

Now let  $a_+ \rightarrow M_2$  and  $a_- \rightarrow M_1$ , keeping the  $b_\pm$  and  $c_\pm$ . Then from (4) follows

$$w(M_2^{(2)}, \bar{M}^{(1)}, t_0) \leq w(M_2^{(2)}, M_H^{(1)}, t_0) + w(M_H^{(2)}, \bar{M}^{(1)}, t_0), \quad (13)$$

and, taking into account (A1):

$$|q|^2 - |p|^2 \leq |p - q|^2. \quad (14)$$

Finally let  $a_+ \rightarrow M_2$ ,  $a_- \rightarrow M_1$ ,  $b_+ \rightarrow M$ ,  $b_- \rightarrow \bar{M}$ , and  $c_+ \rightarrow M_H$ ,  $c_- \rightarrow M_L$ . Then, from (4) it follows that

$$w(M_2^{(2)}, M^{(1)}, t_0) \leq w(M_2^{(2)}, M_H^{(1)}, t_0) + w(M_H^{(2)}, M^{(1)}, t_0), \quad (15)$$

and, in turn,

$$|p|^2 - |q|^2 \leq |p - q|^2. \quad (16)$$

The inequalities (14) and (16) can be merged into

$$||p|^2 - |q|^2| \leq |p - q|^2. \quad (17)$$

The inequalities (12) – (17) represent the full set of the static Wigner inequalities for the oscillations of neutral mesons. The set is obtained from all possible correspondences between the  $a_\pm$ ,  $b_\pm$ , and  $c_\pm$  and the projection of the meson states onto the “directions” of flavour,  $CP$ , and the states with fixed masses and lifetimes. It is obvious that (12) and (17) can

TABLE I: Experimental values of the oscillations and the  $CP$ -violation for the neutral pseudoscalar mesons [31]. The minus sign of  $\Delta\Gamma$  is due to the difference of the definitions between the current work and [31]. The dimensionless variable  $\lambda = \Delta M/\Delta\Gamma$ .

Meson	$\Delta\Gamma$ (MeV)	$\Delta M$ (MeV)	$\tan\beta \equiv  q/p _M^{exp}$	$\lambda$
$B_s^0$	$-6.0 \times 10^{-11}$	$1.2 \times 10^{-8}$	$1.0039 \pm 0.0021$	$-0.2 \times 10^3$
$K^0$	$-7.3 \times 10^{-12}$	$3.5 \times 10^{-12}$	$0.99668 \pm 0.00004$	$-4.8 \times 10^{-1}$
$D^0$	$-2.1 \times 10^{-11}$	$-6.3 \times 10^{-12}$	$0.92^{+0.12}_{-0.09}$	0.3

not be violated simultaneously – one can sum (12) and (17). However there are no physical arguments to prefer (12) over (17) or vice versa.

Using (A1) it can be shown that (12) and (17) reduce to

$$|\cos 2\beta| \pm \cos \zeta \sin(2\beta) \leq 1. \quad (18)$$

Inequality (18) does not contain the unphysical phase  $\alpha$ , as must be the case for any experimentally testable relation between the observables in quantum theory.

We now check whether (12) is violated in systems of neutral pseudoscalar mesons. In Table I the current experimental values of  $|q/p|_M^{exp}$  are shown. For all the neutral mesons  $|q/p|_M^{exp} \approx 1$ , i.e.  $\beta \approx \beta_0 = \pi/4$ . For the  $D$ -mesons,  $|q/p|_D^{exp}$  is not well measured, however within the experimental uncertainties it is consistent with one.

For  $\beta = \beta_0 = \pi/4$  (i.e. without oscillation-induced  $CP$ -violation), (18) reduces to the trivial inequality

$$|\cos \zeta| \leq 1, \quad (19)$$

which is not violated for any value of the phase  $\zeta$ .

However (19) does not mean that the static inequalities (12) and (17) are never violated. Due to the  $CP$ -violation, the angle  $\beta$  is slightly different from  $\beta_0 = \pi/4$ .

For  $K$ -mesons violation of (12) was demonstrated in [22]. In the case at hand, the coefficients  $p$  and  $q$  are defined through the  $CP$ -violation parameter  $\varepsilon$  as:

$$p = \frac{1}{\sqrt{2}} \frac{1 + \varepsilon}{\sqrt{1 + |\varepsilon|^2}} \quad \text{and} \quad q = \frac{1}{\sqrt{2}} \frac{1 - \varepsilon}{\sqrt{1 + |\varepsilon|^2}}.$$

Then (12) becomes

$$|\text{Re}(\varepsilon)| \leq 1. \quad (20)$$



and (17) becomes

$$|\text{Re}(\varepsilon)| \leq |\varepsilon|^2, \quad (21)$$

which corresponds to (16) from [22] if one neglects the corrections  $\sim |\varepsilon|^2$  and the modulus on the left hand side. The inequality (20) is never violated, as  $|\varepsilon| \sim 10^{-3}$  [31], leading to the upper limit  $|\text{Re}(\varepsilon)| \sim 10^{-3}$ . Inequality (21) should be strongly violated, as  $\varphi_\varepsilon = (43.52 \pm 0.05)^\circ$  [31]. However due to the smallness of  $CP$ -violation in neutral  $K$ -mesons, direct experimental tests of (13) and (15) are not possible [22].

### III. TIME-DEPENDENT WIGNER INEQUALITIES FOR THE OSCILLATIONS OF NEUTRAL PSEUDOSCALAR MESONS

We now consider time-dependent Wigner inequalities (5) for neutral pseudoscalar meson systems. Note that the normalization of probability to unity was not used in the derivation of (5). Hence (5) is valid for a unstable particles, whose state vector normalization is time-dependent.

The time evolution of the states  $|M_L\rangle$   $|M_H\rangle$  is given by:

$$\begin{aligned} |M_L(t)\rangle &= e^{-iE_L \Delta t} |M_L\rangle = e^{-im_L \Delta t - \Gamma_L \Delta t/2} |M_L\rangle, \\ |M_H(t)\rangle &= e^{-iE_H \Delta t} |M_H\rangle = e^{-im_H \Delta t - \Gamma_H \Delta t/2} |M_H\rangle, \end{aligned} \quad (22)$$

where  $\Delta t = t - t_0$ . This leads to the evolution of the states  $|M(t)\rangle$   $|\bar{M}(t)\rangle$  as:

$$\begin{cases} |M(t)\rangle = g_+(\Delta t) |M\rangle - e^{i\alpha} \frac{q}{p} g_-(\Delta t) |\bar{M}\rangle \\ |\bar{M}(t)\rangle = g_+(\Delta t) |\bar{M}\rangle - e^{-i\alpha} \frac{p}{q} g_-(\Delta t) |M\rangle \end{cases}.$$

We then obtain the evolution of the states  $|M_1(t)\rangle$  and  $|M_2(t)\rangle$  as:

$$\begin{aligned} |M_1(t)\rangle &= \frac{1}{\sqrt{2}} \left( \left( g_+(\Delta t) - \frac{p}{q} g_-(\Delta t) \right) |M\rangle + e^{i\alpha} \left( g_+(\Delta t) - \frac{q}{p} g_-(\Delta t) \right) |\bar{M}\rangle \right), \\ |M_2(t)\rangle &= \frac{1}{\sqrt{2}} \left( \left( g_+(\Delta t) + \frac{p}{q} g_-(\Delta t) \right) |M\rangle - e^{i\alpha} \left( g_+(\Delta t) + \frac{q}{p} g_-(\Delta t) \right) |\bar{M}\rangle \right), \end{aligned}$$

where  $g_\pm(\tau) = \frac{1}{2} (e^{-iE_H \tau} \pm e^{-iE_L \tau})$ . For the functions  $g_\pm(\tau)$ , the following is satisfied:

$$\begin{aligned} |g_\pm(\tau)|^2 &= \frac{e^{-\Gamma \tau}}{2} \left( \text{ch} \left( \frac{\Delta \Gamma \tau}{2} \right) \pm \cos(\Delta M \tau) \right), \\ g_+^*(\tau) g_-(\tau) &= -\frac{e^{-\Gamma \tau}}{2} \left( \text{sh} \left( \frac{\Delta \Gamma \tau}{2} \right) + i \sin(\Delta M \tau) \right), \end{aligned}$$

where  $\Gamma = (\Gamma_H + \Gamma_L)/2$ . Taking into account initial condition (7), it is possible to write the wave function of the  $M\bar{M}$ -pair at arbitrary time  $t$ :

$$|\Psi(t)\rangle = e^{-i(m_H+m_L)\Delta t} e^{-\Gamma\Delta t} |\Psi(t_0)\rangle. \quad (23)$$

To simplify the subsequent calculations we from now on set  $t_0 = 0$ , so  $\Delta t \equiv t$ .

To demonstrate the advantage of (5) over (4), we first consider the case of  $\beta = \beta_0 = \pi/4$ , when there is no oscillation-induced  $CP$ -violation. The static inequality (4) becomes the never-violated inequality (19). One can obtain a significant simplification by considering the time-dependent (5) with the additional condition  $\cos\zeta = \pm 1$ . If one neglects  $CP$ -violation, then for  $K$ -mesons,  $\left(\frac{q}{p}\right)_K = \frac{1-\epsilon}{1+\epsilon} \approx 1$ , so  $\cos\zeta_K = 1$ . For  $B_q$ -mesons the effective Hamiltonian of the oscillations is proportional to  $(V_{tb}V_{tq}^*)^2$  [32]. Then:

$$\left(\frac{q}{p}\right)_{B_q} = -\frac{H_{21}}{\sqrt{H_{12}H_{21}}} \approx -\left(\frac{V_{tb}^*V_{tq}}{|V_{tb}^*V_{tq}|}\right)^2 = -1,$$

hence  $\cos\zeta_{B_q} = -1$ . For  $D$ -mesons the experimental results of BaBar [33] and Belle [34] are in accordance with the assumption  $\cos\zeta_D = 1$ , in which case the condition  $\cos\zeta = \pm 1$  is well justified.

Table II shows time-dependent Wigner inequalities for all possible correspondences between the dichotomic variables  $a_{\pm}$ ,  $b_{\pm}$ ,  $c_{\pm}$  and the projections of meson states onto the “directions” of flavour,  $CP$ , and states with fixed masses and lifetimes.

All calculations are performed using formulas (A1) and (A2) with the approximation  $\beta = \beta_0 = \pi/4$  and  $\cos\zeta = \pm 1$ . The experimental values of the oscillation parameters shown in Table I, and numerical estimates of the  $\cos\zeta_M$  suggest the optimal choice of sets N5 and N6 from Table II for studying the violation of (5) in  $K$ - and  $D$ -mesons. For studying the violation of (5) in oscillations of neutral  $B_{d,s}$ -mesons one should choose sets N7 and N8.

#### IV. $CP$ -VIOLATION EFFECTS INFLUENCING THE VIOLATION OF THE TIME-DEPENDENT WIGNER INEQUALITIES

We now take into account all the  $CP$ -violation effects, i.e. the case when  $\beta \neq \beta_0$ , and  $\cos\zeta_M \neq \pm 1$ . Then for the various sets from Table II, the substitution of (A1) and (A2) into the time-dependent Wigner inequalities (5) results in eight inequalities. They can be

TABLE II: Time-dependent Wigner inequalities (5) for neutral pseudoscalar mesons with the approximation  $\beta = \beta_0 = \pi/4$  and  $\cos \zeta = \pm 1$ . All possible correspondences between the dichotomic variables  $a_{\pm}$ ,  $b_{\pm}$ ,  $c_{\pm}$  and the projections of the meson states onto the “directions” of flavour,  $CP$ , and the states with fixed masses and lifetimes are shown.

N	Correspondence of the variables	Time-dependent Wigner inequalities	Violation conditions
1	$a_+ \rightarrow M_1, b_+ \rightarrow \bar{M}, c_+ \rightarrow M_H,$ $a_- \rightarrow M_2, b_- \rightarrow M, c_- \rightarrow M_L$	$1 \leq e^{-\Delta\Gamma t}$ when $\cos \zeta = -1$	if $\Delta\Gamma \geq 0$
2	$a_+ \rightarrow M_1, b_+ \rightarrow M, c_+ \rightarrow M_H,$ $a_- \rightarrow M_2, b_- \rightarrow \bar{M}, c_- \rightarrow M_L$	$1 \leq e^{-\Delta\Gamma t}$ when $\cos \zeta = -1$	if $\Delta\Gamma \geq 0$
3	$a_+ \rightarrow M_2, b_+ \rightarrow \bar{M}, c_+ \rightarrow M_H,$ $a_- \rightarrow M_1, b_- \rightarrow M, c_- \rightarrow M_L$	$1 \leq e^{-\Delta\Gamma t}$ when $\cos \zeta = +1$	if $\Delta\Gamma \geq 0$
4	$a_+ \rightarrow M_2, b_+ \rightarrow M, c_+ \rightarrow M_H,$ $a_- \rightarrow M_1, b_- \rightarrow \bar{M}, c_- \rightarrow M_L$	$1 \leq e^{-\Delta\Gamma t}$ when $\cos \zeta = +1$	if $\Delta\Gamma \geq 0$
5	$a_+ \rightarrow M_1, b_+ \rightarrow \bar{M}, c_+ \rightarrow M_L,$ $a_- \rightarrow M_2, b_- \rightarrow M, c_- \rightarrow M_H$	$1 \leq e^{\Delta\Gamma t}$ when $\cos \zeta = +1$	if $\Delta\Gamma \leq 0$
6	$a_+ \rightarrow M_1, b_+ \rightarrow M, c_+ \rightarrow M_L,$ $a_- \rightarrow M_2, b_- \rightarrow \bar{M}, c_- \rightarrow M_H$	$1 \leq e^{\Delta\Gamma t}$ when $\cos \zeta = +1$	if $\Delta\Gamma \leq 0$
7	$a_+ \rightarrow M_2, b_+ \rightarrow \bar{M}, c_+ \rightarrow M_L,$ $a_- \rightarrow M_1, b_- \rightarrow M, c_- \rightarrow M_H$	$1 \leq e^{\Delta\Gamma t}$ when $\cos \zeta = -1$	if $\Delta\Gamma \leq 0$
8	$a_+ \rightarrow M_2, b_+ \rightarrow M, c_+ \rightarrow M_L,$ $a_- \rightarrow M_1, b_- \rightarrow \bar{M}, c_- \rightarrow M_H$	$1 \leq e^{\Delta\Gamma t}$ when $\cos \zeta = -1$	if $\Delta\Gamma \leq 0$

reduced to:

$$1 \leq R_N(x, r, \zeta, \lambda). \quad (24)$$

The functions  $R_N$  depend on the dimensionless variables  $x = \Delta\Gamma t$ ,  $\lambda = \Delta^M/\Delta\Gamma$ , the absolute value  $r$ , and the phase  $\zeta$  of the ratio  $q/p$ . The experimental values of  $\Delta\Gamma$  are less or equal to 0 for  $K^-$ ,  $D^-$ , and  $B_s^-$ -mesons, thus we consider only the functions  $R_5 - R_8$ . Analytical expressions for these functions are given in Appendix B. Numerical values of the parameters used for the analysis of the inequalities (24) are given in Table I.

We start with the system of neutral kaons. For  $K$ -mesons the absolute values and phase of the  $CP$  violation parameter  $\varepsilon$  are known with quite high precision. Hence  $r$  and  $\zeta$  are also well defined. In FIG. 1 the functions  $R_5(x, r, \zeta, \lambda)$  and  $R_6(x, r, \zeta, \lambda)$  are shown. For kaons these functions are almost identical for the experimentally allowed values of  $r$  and  $\zeta$  (e.g. with  $r = 0.997$  and  $\zeta = -0.18^\circ$ , which are used in FIG 1). The top scale corresponds to the variable  $ct$  (the decay length) in mm. The bottom scale corresponds to time measured in units of the average kaon lifetimes  $z = \frac{1}{2}(\Gamma_H + \Gamma_L) t = \Gamma t$ . Time  $t$  is calculated in the  $K$ -meson rest frame. In FIG. 2 the functions  $R_{5,6}(x, r, \zeta, \lambda)$  are shown for  $z \leq 3$ , i.e. in the most experimentally accessible area.

Time-dependent inequalities (24) are violated when  $R_{5,6}(x, r, \zeta, \lambda) < 1$ . FIG. 1, shows that for the  $K$ -mesons this violation occurs when  $z \lesssim 5.5$ , i.e., in the experimentally accessible area. For  $z \gtrsim 5.5$  the inequalities (24) are not violated. This interesting effect may be understood if one makes an expansion of the functions  $R_N(x, r, \zeta, \lambda)$  by small parameters  $\Delta r = r - 1$  and  $\zeta$ . As an example we obtain the expansion of the function  $R_5(x, r, \zeta, \lambda)$  to second order. We consider  $\Delta r$  and  $\zeta$  to be of the same order of magnitude. Then:

$$\begin{aligned} R_5(x, r, \zeta, \lambda) \approx & \frac{1}{2}(e^x + 1)(1 - \Delta r) + \\ & + \left( 3 \operatorname{ch}^2\left(\frac{x}{2}\right) + \frac{3}{2} \operatorname{sh}\left(\frac{x}{2}\right) \operatorname{ch}\left(\frac{x}{2}\right) - 2 \operatorname{ch}\left(\frac{x}{2}\right) \cos(\lambda x) \right) (\Delta r)^2 + \\ & + \left( \frac{3}{2} \operatorname{ch}^2\left(\frac{x}{2}\right) - \operatorname{ch}\left(\frac{x}{2}\right) \cos(\lambda x) \right) \zeta^2 - \operatorname{ch}\left(\frac{x}{2}\right) \sin(\lambda x) \zeta \Delta r. \end{aligned}$$

At zeroth order in  $\Delta r$  and  $\zeta$ , which corresponds to the absence of  $CP$ -violation, the effect of restoration of the inequality (24) at large values of  $t$  (or  $z$ ) does not appear. That is, this effect is fully determined by the  $CP$ -violation. A first order expansion is also not enough. The effect appears when, at a particular value of  $z$  the second order contribution begins to be comparable to the previous orders and the expansion is no longer valid. Note that in the range of low  $z$ , which is the most experimentally interesting, the approximation from Table

II is thus proved to be acceptable. Similar properties can be observed in the expansion of the function  $R_6(x, r, \zeta, \lambda)$ .

For neutral  $D$ -mesons the situation is very similar to the one with  $K$ -mesons. In order to study the violation of (24), it is necessary to consider the dependence on  $z$  of the functions  $R_{5,6}(x, r, \zeta, \lambda)$ . However, unlike the  $K$ -meson case, the parameters  $r$  and  $\zeta$  for  $D$ -mesons are not well fixed from experiment. The evolution of the set of parameters  $r$  and  $\zeta$ , which violate (24), with  $z$  (or  $ct$ ) is shown in FIG. 3. Gray areas correspond to the function  $R_5(x, r, \zeta, \lambda)$ . Hatched areas correspond to  $R_6(x, r, \zeta, \lambda)$ . The area of experimentally allowed values of the parameters  $r$  and  $\zeta$  is contained within the rectangle. At  $t = 0$  the areas do not intersect, but have only a common point at  $r = 1$  and  $\zeta = 0^\circ$ . As  $t \rightarrow +\infty$  both areas shrink to the point  $r = 1$  and  $\zeta = 0^\circ$ , corresponding to the results of Table II.

From FIG. 3 one can see that with  $r > 1$  in the limit  $t \rightarrow 0$ , inequality (24) is only violated for the function  $R_5(x, r, \zeta, \lambda)$ . For  $r < 1$  in the limit  $t \rightarrow 0$  the violation only happens for the function  $R_6(x, r, \zeta, \lambda)$ . This statement is illustrated in FIG. 4. For the function  $R_5(x, r, \zeta, \lambda)$ ,  $r$  is set to 1.1. For the function  $R_6(x, r, \zeta, \lambda)$ ,  $r$  is set to 0.9. The value of  $\zeta$  in both cases is set to  $-10^\circ$ . In analogy with the case of  $K$ -mesons, there is a restoration of the inequalities (24), but with a higher value of  $z$ ,  $\sim 280$ , which is beyond experimental reach. Maximal violation of (24) also happens at  $z \sim 150$ . From FIG. 5 one can see that in the experimentally allowed area  $z \leq 3$  the violation of (24) is less than 10%.

The difference in behaviour of the functions  $R_{5,6}(x, r, \zeta, \lambda)$  for  $K$ - and  $D$ -mesons is linked to the value of the ratio  $|\Delta\Gamma|/\Gamma$ , which sets the scale of the horizontal axis. For  $K$ -mesons,  $\left(\frac{|\Delta\Gamma|}{\Gamma}\right)_K \approx 2$ , while for  $D$ -mesons this parameter is smaller by almost two orders of magnitude,  $\left(\frac{|\Delta\Gamma|}{\Gamma}\right)_D \approx 10^{-2}$ .

As was pointed out above, the study of the violation of (24) for  $B_s$ -meson systems requires the functions  $R_{7,8}(x, r, \zeta, \lambda)$ . In FIG. 6 we show how the areas of violation of (24) depend on  $z$  or  $ct$  for the functions  $R_7(x, r, \zeta, \lambda)$  (gray area) and  $R_8(x, r, \zeta, \lambda)$  (hatched area). The vertical band shows the experimentally allowed values of  $r$  and  $\zeta$ . For  $t = 0$  the areas have a single point of intersection,  $r = 1$  and  $\zeta = 180^\circ$ . As  $t \rightarrow +\infty$  they shrink to a point at  $(1, 180^\circ)$ . Unlike the case for  $D$ -mesons, the areas of violation for the  $B_s$ -mesons do not evolve monotonically. This is due to the oscillations of  $B_s$ -mesons, which play an important role here.

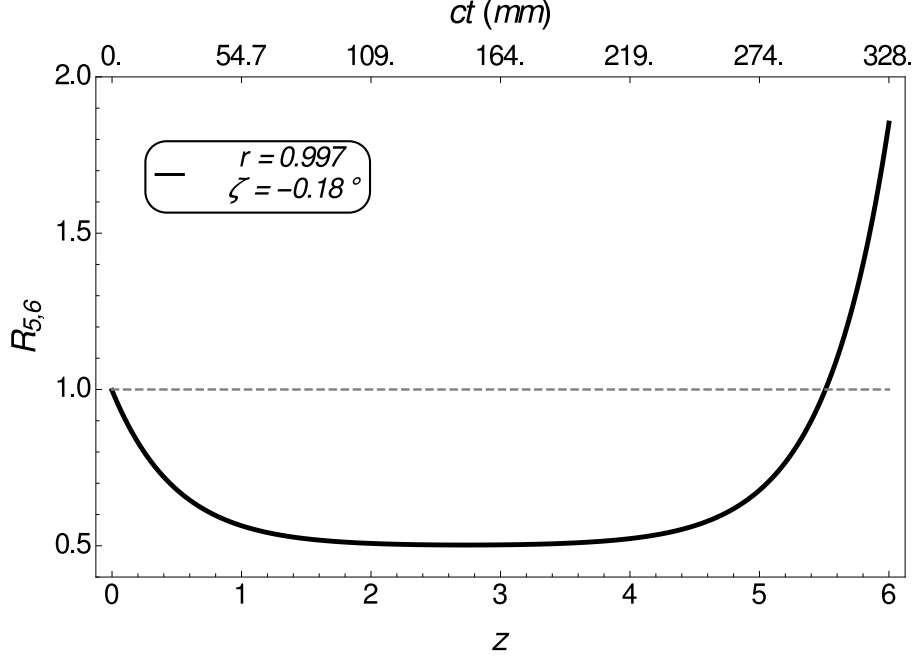


FIG. 1: Functions  $R_{5,6}(x, r, \zeta, \lambda)$  for neutral  $K$ -mesons (both functions are almost juxtaposed due to the high accuracy of the  $CP$ -violation parameter  $\varepsilon$ ). The scale at the top corresponds to the variable  $ct$  (mm); the bottom scale – to the time in units of the average lifetime  $z = (\Gamma_H + \Gamma_L) t / 2 = \Gamma t$ , where  $t$  is kaon rest frame time.

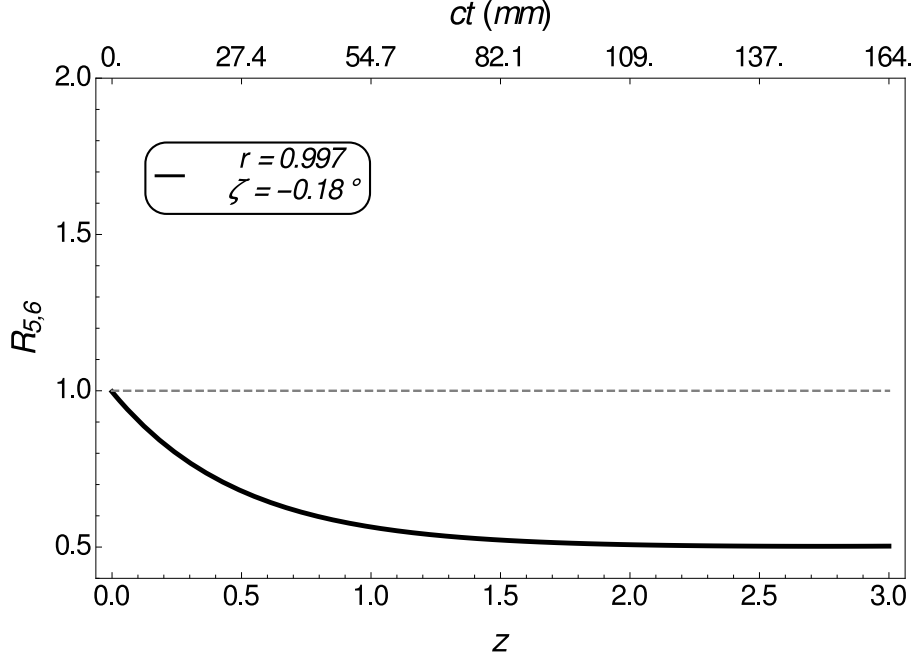


FIG. 2: Functions  $R_{5,6}(x, r, \zeta, \lambda)$  for neutral  $K$ -mesons for  $z \leq 3$ .

It is experimentally established that for  $B_s$ -mesons,  $r > 1$ . Hence for  $z \rightarrow 0$  only the function  $R_7(x, r, \zeta, \lambda)$  violates the inequality (24). However in FIG. 8 one observes that for  $z \gtrsim \Gamma/(2\Delta M) \sim 1/2$  the function  $R_8(x, r, \zeta, \lambda)$  also begins to violate (24). For the numerical simulation, the following values of the parameters were used:  $r = 1.004$  and  $\zeta = 185^\circ$ . The maximum violation of (24) is reached in the area  $z \sim 20$ . At  $z \sim 40$  the inequalities are not violated, as in  $D$ -meson systems. Due to the high value of  $z$  this effect is not experimentally reachable. As  $\left(\frac{|\Delta\Gamma|}{\Gamma}\right)_{B_z} \approx 0.13$ , the corresponding values of  $z$  are intermediate between the ones for  $K^-$  and  $D^-$ -mesons.

## Conclusions

Using the oscillations of neutral pseudoscalar mesons we demonstrate the advantages of the time-dependent Wigner inequality (5) over the static inequality (4). Eight new time-dependent inequalities (24) were obtained. They can be violated by proper choices of  $\Delta\Gamma$  and  $q/p$  for  $K^-$ ,  $D^-$  and  $B_s$ -mesons. Relaxation of the obtained inequalities at high values of the variable  $z$ , is found. This effect is governed explicitly by the  $CP$ -violation parameters of the considered systems. The inequalities (24) may be tested at contemporary high-energy physics experiments

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## Appendix A: Probabilities required for the time-dependent Wigner inequalities

In this appendix we summarize all the probabilities that are required to obtain the static (4) and time-dependent (5) Wigner inequalities for the correlated systems of neutral pseudoscalar mesons.

In the framework of quantum theory, using the normalization condition (6) and the initial condition (7), one can obtain the following expressions for the time-independent probabilities:

$$\begin{aligned}
w(M_1^{(2)}, \bar{M}^{(1)}, t_0) &= \left| \langle M_1^{(2)} | \langle \bar{M}^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{4} \equiv \frac{1}{4} (|p|^2 + |q|^2); \\
w(M_1^{(2)}, M^{(1)}, t_0) &= \left| \langle M_1^{(2)} | \langle M^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{4} \equiv \frac{1}{4} (|p|^2 + |q|^2); \\
w(M_2^{(2)}, \bar{M}^{(1)}, t_0) &= \left| \langle M_2^{(2)} | \langle \bar{M}^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{4} \equiv \frac{1}{4} (|p|^2 + |q|^2); \\
w(M_2^{(2)}, M^{(1)}, t_0) &= \left| \langle M_2^{(2)} | \langle M^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{4} \equiv \frac{1}{4} (|p|^2 + |q|^2); \tag{A1} \\
w(M_1^{(2)}, M_H^{(1)}, t_0) &= \left| \langle M_1^{(2)} | \langle M_H^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{4} |p + q|^2 = \frac{1}{4} (1 + \cos \zeta \sin(2\beta)); \\
w(M_2^{(2)}, M_H^{(1)}, t_0) &= \left| \langle M_2^{(2)} | \langle M_H^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{4} |p - q|^2 = \frac{1}{4} (1 - \cos \zeta \sin(2\beta)); \\
w(M_1^{(2)}, M_L^{(1)}, t_0) &= \left| \langle M_1^{(2)} | \langle M_L^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{4} |p - q|^2 = \frac{1}{4} (1 - \cos \zeta \sin(2\beta)); \\
w(M_2^{(2)}, M_L^{(1)}, t_0) &= \left| \langle M_2^{(2)} | \langle M_L^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{4} |p + q|^2 = \frac{1}{4} (1 + \cos \zeta \sin(2\beta)); \\
w(M_H^{(2)}, \bar{M}^{(1)}, t_0) &= \left| \langle M_H^{(2)} | \langle \bar{M}^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{2} |p|^2 = \frac{1}{2} \cos^2 \beta; \\
w(M_H^{(2)}, M^{(1)}, t_0) &= \left| \langle M_H^{(2)} | \langle M^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{2} |q|^2 = \frac{1}{2} \sin^2 \beta; \\
w(M_L^{(2)}, \bar{M}^{(1)}, t_0) &= \left| \langle M_L^{(2)} | \langle \bar{M}^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{2} |p|^2 = \frac{1}{2} \cos^2 \beta; \\
w(M_L^{(2)}, M^{(1)}, t_0) &= \left| \langle M_L^{(2)} | \langle M^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{2} |q|^2 = \frac{1}{2} \sin^2 \beta.
\end{aligned}$$

To obtain the inequality (5) for correlated pairs of mesons  $M\bar{M}$ , we need the values of the following time-dependent probabilities ( $t_0 = 0$  below):

$$\begin{aligned}
w(M_1(0) \rightarrow M_1(t)) &= |\langle M_1(t) | M_1 \rangle|^2 = \left| g_+(t) - \frac{1}{2} \left( \frac{q}{p} + \frac{p}{q} \right) g_-(t) \right|^2; \\
w(M_2(0) \rightarrow M_1(t)) &= |\langle M_1(t) | M_2 \rangle|^2 = \left| \frac{1}{2} \left( \frac{q}{p} - \frac{p}{q} \right) g_-(t) \right|^2; \\
w(M_2(0) \rightarrow M_2(t)) &= |\langle M_2(t) | M_2 \rangle|^2 = \left| g_+(t) + \frac{1}{2} \left( \frac{q}{p} + \frac{p}{q} \right) g_-(t) \right|^2;
\end{aligned}$$



$$\begin{aligned}
w(M_1(0) \rightarrow M_2(t)) &= |\langle M_2(t) | M_1 \rangle|^2 = \left| \frac{1}{2} \left( \frac{q}{p} - \frac{p}{q} \right) g_-(t) \right|^2; \\
w(\bar{M}(0) \rightarrow \bar{M}(t)) &= |\langle \bar{M}(t) | \bar{M} \rangle|^2 = |g_+(t)|^2; \\
w(M(0) \rightarrow \bar{M}(t)) &= |\langle \bar{M}(t) | M \rangle|^2 = \left| \frac{p}{q} g_-(t) \right|^2; \\
w(M(0) \rightarrow M(t)) &= |\langle M(t) | M \rangle|^2 = |g_+(t)|^2; \\
w(\bar{M}(0) \rightarrow M(t)) &= |\langle M(t) | \bar{M} \rangle|^2 = \left| \frac{q}{p} g_-(t) \right|^2; \\
w(M_1^{(2)}, \bar{M}^{(1)}, t) &= \left| \langle M_1^{(2)} | \langle \bar{M}^{(1)} | \Psi(t) \rangle \right|^2 = \frac{1}{4} e^{-2\Gamma t}; \\
w(M_1^{(2)}, M^{(1)}, t) &= w(M_2^{(2)}, \bar{M}^{(1)}, t) = w(M_2^{(2)}, M^{(1)}, t) = \frac{1}{4} e^{-2\Gamma t}.
\end{aligned} \tag{A2}$$

## Appendix B: Functions $R_N$

The functions  $R_N(x, r, \zeta, \lambda)$  depend on the dimensionless parameters  $x = \Delta\Gamma t$ ,  $\lambda = \Delta M/\Delta\Gamma$ , the absolute value  $r$ , and the phase  $\zeta$  of  $q/p$ . In the most general way these functions can be written as:

$$\begin{aligned}
R_N(x, r, \zeta, \lambda) &= f_0^{(N)} + f_{0c}^{(N)} \cos(2\lambda x) + f_{0s}^{(N)} \sin(2\lambda x) + f_{1s}^{(N)} \text{sh}(x) + f_{1c}^{(N)} \text{ch}(x) + \\
&+ f_{2sc}^{(N)} \text{sh}\left(\frac{x}{2}\right) \cos(\lambda x) + f_{2cc}^{(N)} \text{ch}\left(\frac{x}{2}\right) \cos(\lambda x) + f_{2cs}^{(N)} \text{ch}\left(\frac{x}{2}\right) \sin(\lambda x),
\end{aligned}$$

where we explicitly show the dependence of  $R_N$  on  $x$  and  $\lambda$ . The coefficients  $f^{(N)}(r, \zeta)$  are:

for  $N = 5$

$$\begin{aligned}
f_0^{(5)}(r, \zeta) &= \frac{5r^6 - 4r^5 \cos(\zeta) + 7r^4 - 2r^3 \cos(\zeta) - 2r^3 \cos(3\zeta) + 3r^2 + 1}{8r^4 (r^2 + 1)}; \\
f_{0c}^{(5)}(r, \zeta) &= -\frac{(r-1)(r+1)(r^6 - r^4 + 2r^3 \cos(\zeta) - 2r^3 \cos(3\zeta) - r^2 + 1)}{16r^4 (r^2 + 1)}; \\
f_{0s}^{(5)}(r, \zeta) &= -\frac{(r-1)^2(r+1)^2 \sin(\zeta)(r^2 - 2r \cos(\zeta) + 1)}{8r^3 (r^2 + 1)}; \\
f_{1s}^{(5)}(r, \zeta) &= \frac{\cos(\zeta)(r^4 - 2r^3 \cos(\zeta) + 6r^2 - 2r \cos(\zeta) + 1)}{8r^3}; \\
f_{1c}^{(5)}(r, \zeta) &= \frac{r^6 + 7r^4 - 6r^3 \cos(\zeta) - 2r^3 \cos(3\zeta) + 7r^2 + 1}{16r^4}; \\
f_{2sc}^{(5)}(r, \zeta) &= \frac{(r-1)(r+1) \cos(\zeta)(r^2 - 2r \cos(\zeta) + 1)}{4r^3}; \\
f_{2cc}^{(5)}(r, \zeta) &= -\frac{r^6 + 4r^5 \cos(\zeta) - 5r^4 - 2r^3 \cos(\zeta) - 2r^3 \cos(3\zeta) + 3r^2 + 1}{4r^4 (r^2 + 1)}; \\
f_{2cs}^{(5)}(r, \zeta) &= -\frac{(r-1)(r+1) \sin(\zeta)(r^4 - 2r^3 \cos(\zeta) + 6r^2 - 2r \cos(\zeta) + 1)}{4r^3 (r^2 + 1)};
\end{aligned}$$

for  $N = 6$

$$\begin{aligned}
f_0^{(6)}(r, \zeta) &= \frac{r^6 + 3r^4 - 2r^3 \cos(\zeta) - 2r^3 \cos(3\zeta) + 7r^2 - 4r \cos(\zeta) + 5}{8(r^2 + 1)}; \\
f_{0c}^{(6)}(r, \zeta) &= \frac{(r-1)(r+1)(r^6 - r^4 + 2r^3 \cos(\zeta) - 2r^3 \cos(3\zeta) - r^2 + 1)}{16r^2(r^2 + 1)}; \\
f_{0s}^{(6)}(r, \zeta) &= \frac{(r-1)^2(r+1)^2 \sin(\zeta)(r^2 - 2r \cos(\zeta) + 1)}{8r(r^2 + 1)}; \\
f_{1s}^{(6)}(r, \zeta) &= \frac{\cos(\zeta)(r^4 - 2r^3 \cos(\zeta) + 6r^2 - 2r \cos(\zeta) + 1)}{8r}; \\
f_{1c}^{(6)}(r, \zeta) &= \frac{r^6 + 7r^4 - 6r^3 \cos(\zeta) - 2r^3 \cos(3\zeta) + 7r^2 + 1}{16r^2}; \\
f_{2sc}^{(6)}(r, \zeta) &= -\frac{(r-1)(r+1) \cos(\zeta)(r^2 - 2r \cos(\zeta) + 1)}{4r}; \\
f_{2cc}^{(6)}(r, \zeta) &= -\frac{r^6 + 3r^4 - 2r^3 \cos(\zeta) - 2r^3 \cos(3\zeta) - 5r^2 + 4r \cos(\zeta) + 1}{4(r^2 + 1)}; \\
f_{2cs}^{(6)}(r, \zeta) &= -\frac{(r-1)(r+1) \sin(\zeta)(r^4 - 2r^3 \cos(\zeta) + 6r^2 - 2r \cos(\zeta) + 1)}{4r(r^2 + 1)};
\end{aligned}$$

for  $N = 7$

$$\begin{aligned}
f_0^{(7)}(r, \zeta) &= \frac{5r^6 + 4r^5 \cos(\zeta) + 7r^4 + 2r^3 \cos(\zeta) + 2r^3 \cos(3\zeta) + 3r^2 + 1}{8r^4(r^2 + 1)}; \\
f_{0c}^{(7)}(r, \zeta) &= -\frac{(r-1)(r+1)(r^6 - r^4 - 2r^3 \cos(\zeta) + 2r^3 \cos(3\zeta) - r^2 + 1)}{16r^4(r^2 + 1)}; \\
f_{0s}^{(7)}(r, \zeta) &= \frac{(r-1)^2(r+1)^2 \sin(\zeta)(r^2 + 2r \cos(\zeta) + 1)}{8r^3(r^2 + 1)}; \\
f_{1s}^{(7)}(r, \zeta) &= -\frac{\cos(\zeta)(r^4 + 2r^3 \cos(\zeta) + 6r^2 + 2r \cos(\zeta) + 1)}{8r^3}; \\
f_{1c}^{(7)}(r, \zeta) &= \frac{r^6 + 7r^4 + 6r^3 \cos(\zeta) + 2r^3 \cos(3\zeta) + 7r^2 + 1}{16r^4}; \\
f_{2sc}^{(7)}(r, \zeta) &= -\frac{(r-1)(r+1) \cos(\zeta)(r^2 + 2r \cos(\zeta) + 1)}{4r^3}; \\
f_{2cc}^{(7)}(r, \zeta) &= -\frac{r^6 - 4r^5 \cos(\zeta) - 5r^4 + 2r^3 \cos(\zeta) + 2r^3 \cos(3\zeta) + 3r^2 + 1}{4r^4(r^2 + 1)}; \\
f_{2cs}^{(7)}(r, \zeta) &= \frac{(r-1)(r+1) \sin(\zeta)(r^4 + 2r^3 \cos(\zeta) + 6r^2 + 2r \cos(\zeta) + 1)}{4r^3(r^2 + 1)}.
\end{aligned}$$

for  $N = 8$

$$\begin{aligned}
f_0^{(8)}(r, \zeta) &= \frac{r^6 + 3r^4 + 2r^3 \cos(\zeta) + 2r^3 \cos(3\zeta) + 7r^2 + 4r \cos(\zeta) + 5}{8(r^2 + 1)}; \\
f_{0c}^{(8)}(r, \zeta) &= \frac{(r-1)(r+1)(r^6 - r^4 - 2r^3 \cos(\zeta) + 2r^3 \cos(3\zeta) - r^2 + 1)}{16r^2(r^2 + 1)};
\end{aligned}$$

$$\begin{aligned}
f_{0s}^{(8)}(r, \zeta) &= -\frac{(r-1)^2(r+1)^2 \sin(\zeta) (r^2 + 2r \cos(\zeta) + 1)}{8r (r^2 + 1)}; \\
f_{1s}^{(8)}(r, \zeta) &= -\frac{\cos(\zeta) (r^4 + 2r^3 \cos(\zeta) + 6r^2 + 2r \cos(\zeta) + 1)}{8r}; \\
f_{1c}^{(8)}(r, \zeta) &= \frac{r^6 + 7r^4 + 6r^3 \cos(\zeta) + 2r^3 \cos(3\zeta) + 7r^2 + 1}{16 r^2}; \\
f_{2sc}^{(8)}(r, \zeta) &= \frac{(r-1)(r+1) \cos(\zeta) (r^2 + 2r \cos(\zeta) + 1)}{4r}; \\
f_{2cc}^{(8)}(r, \zeta) &= -\frac{r^6 + 3r^4 + 2r^3 \cos(\zeta) + 2r^3 \cos(3\zeta) - 5r^2 - 4r \cos(\zeta) + 1}{4 (r^2 + 1)}; \\
f_{2cs}^{(8)}(r, \zeta) &= \frac{(r-1)(r+1) \sin(\zeta) (r^4 + 2r^3 \cos(\zeta) + 6r^2 + 2r \cos(\zeta) + 1)}{4r (r^2 + 1)}.
\end{aligned}$$


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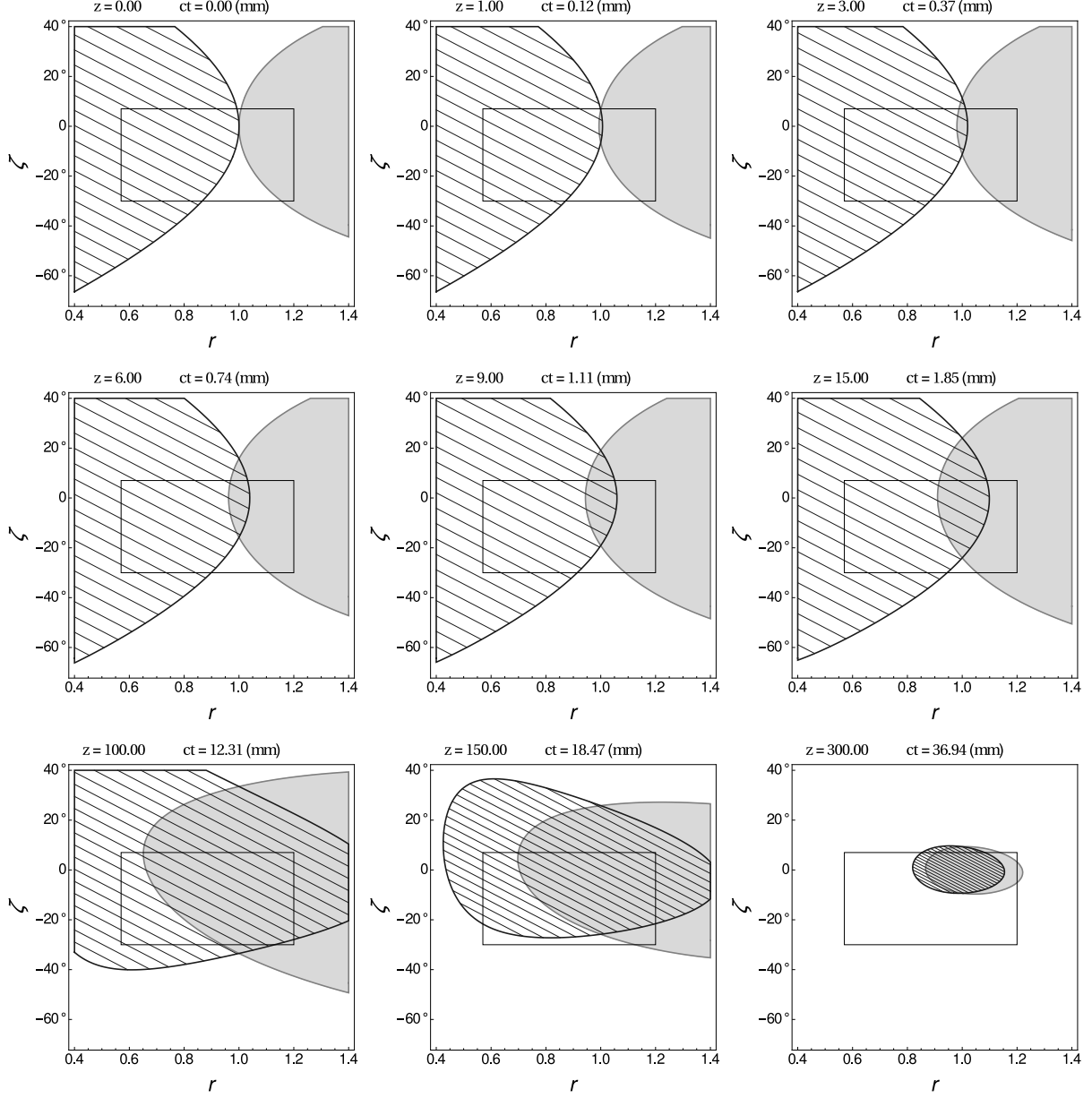


FIG. 3: Areas of violation of the inequalities (24) for  $D$ -mesons for the functions  $R_{5,6}(x, r, \zeta, \lambda)$  in the parameter plane of  $r$  and  $\zeta$  ( $\zeta$  is measured in degrees) depending on  $z$  or  $ct$ . The gray areas correspond to the violation of the function  $R_5(x, r, \zeta, \lambda)$ , while the hatched areas correspond to the function  $R_6(x, r, \zeta, \lambda)$ . The experimentally allowed area of  $r$  and  $\zeta$  is contained within the rectangle.

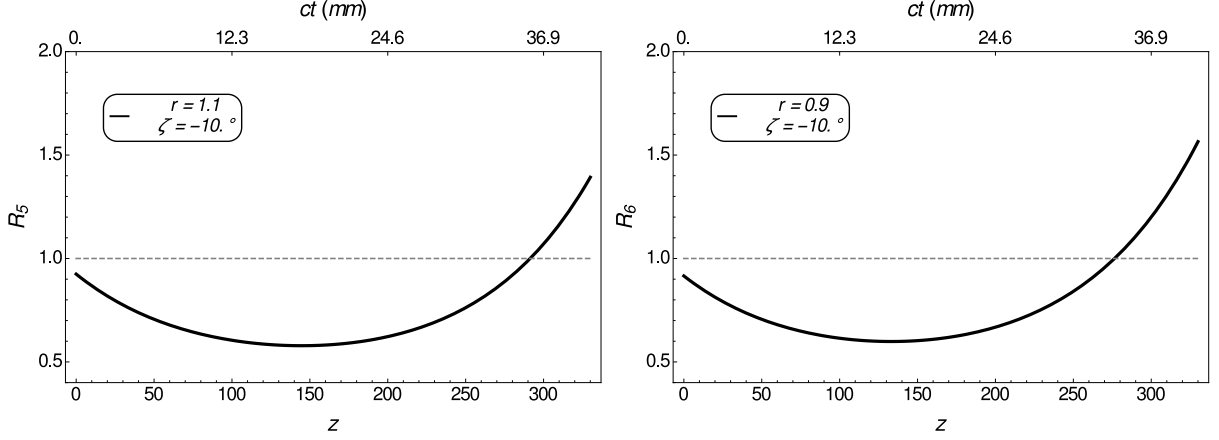


FIG. 4: Functions  $R_{5,6}(x, r, \zeta, \lambda)$  for neutral  $D$ -mesons. The scale at the top corresponds to  $ct$  (mm); the bottom scale corresponds to the time in units of the average lifetime  $z = (\Gamma_H + \Gamma_L) t / 2 = \Gamma t$ , where  $t$  is calculated in the  $D$ -meson rest frame. One can see that with the proper choice of the functions  $R_N$  for  $r > 1$  and  $r < 1$  the time-dependent Wigner inequalities (24) are violated in the whole experimentally accessible range of  $z$ .

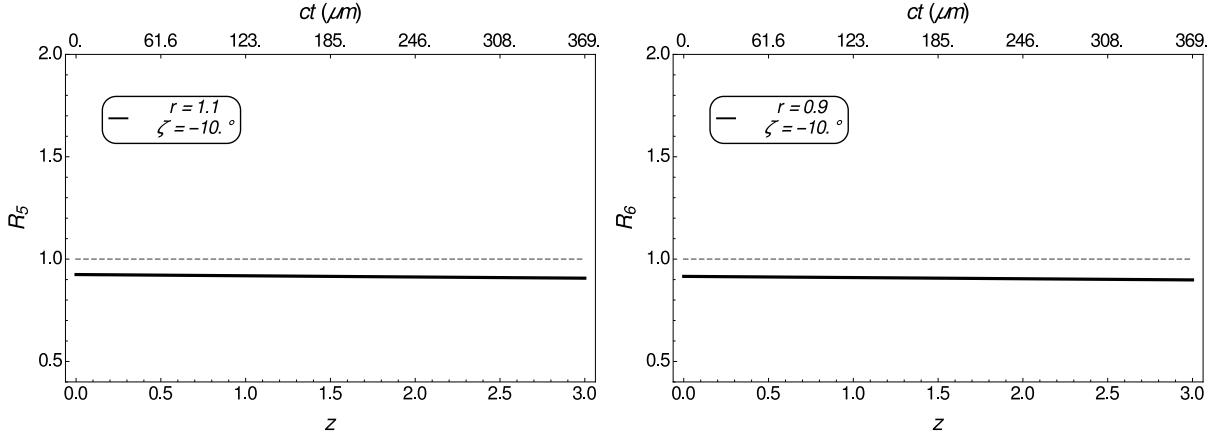


FIG. 5: Functions  $R_{5,6}(x, r, \zeta, \lambda)$  for neutral  $D$ -mesons in the area  $z \leq 3$ . The  $ct$  here is measured in microns ( $\mu\text{m}$ ).

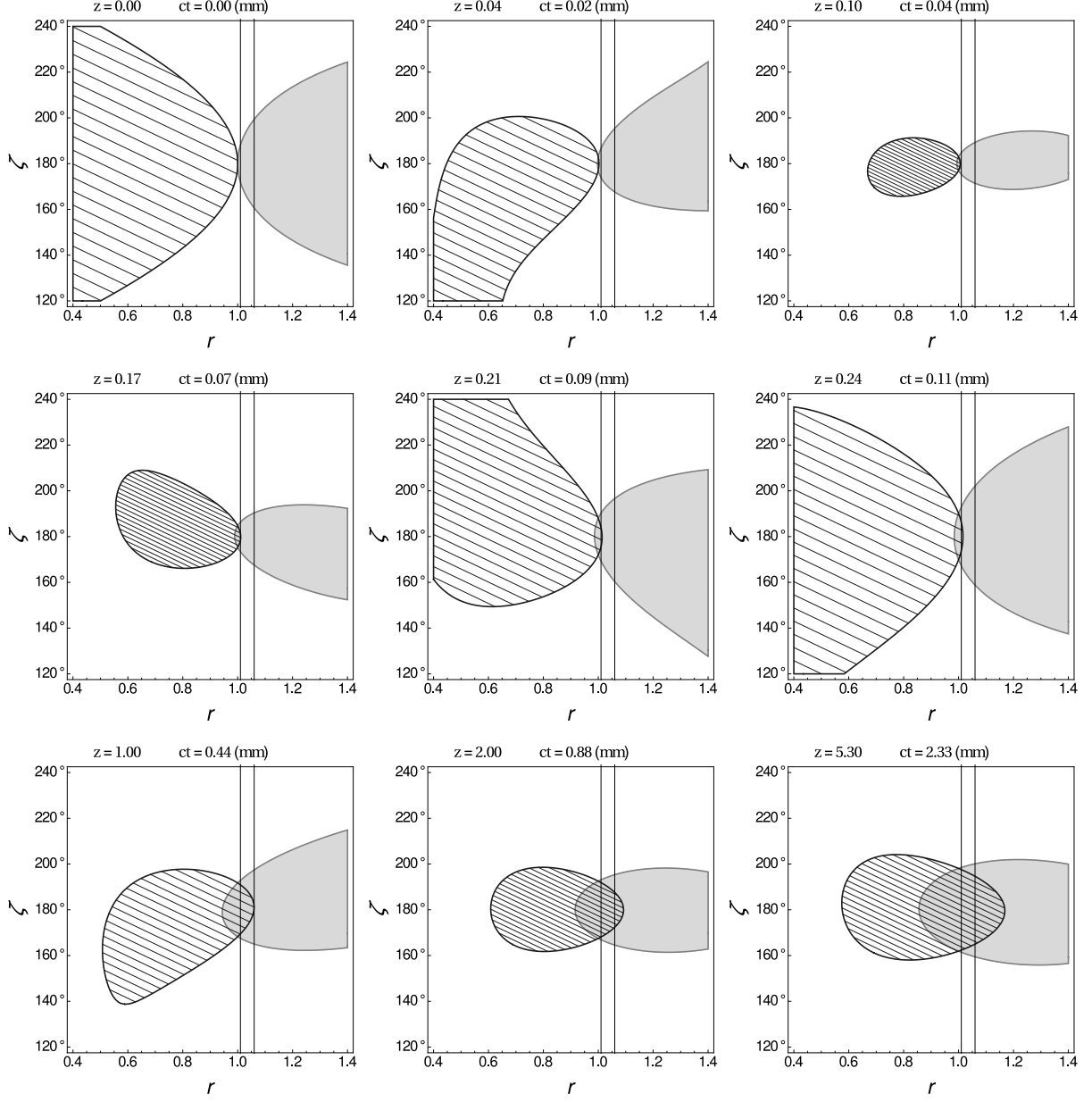


FIG. 6: Areas of the violation of (24) for  $B_s$ -mesons for the functions  $R_{7,8}(x, r, \zeta, \lambda)$  in the  $r$ - $\zeta$  plane ( $\zeta$  is measured in degrees). The gray areas correspond to the function  $R_7(x, r, \zeta, \lambda)$  the hatched areas correspond to the function  $R_8(x, r, \zeta, \lambda)$ . The vertical band corresponds to the experimentally allowed area of  $r$  and  $\zeta$ .

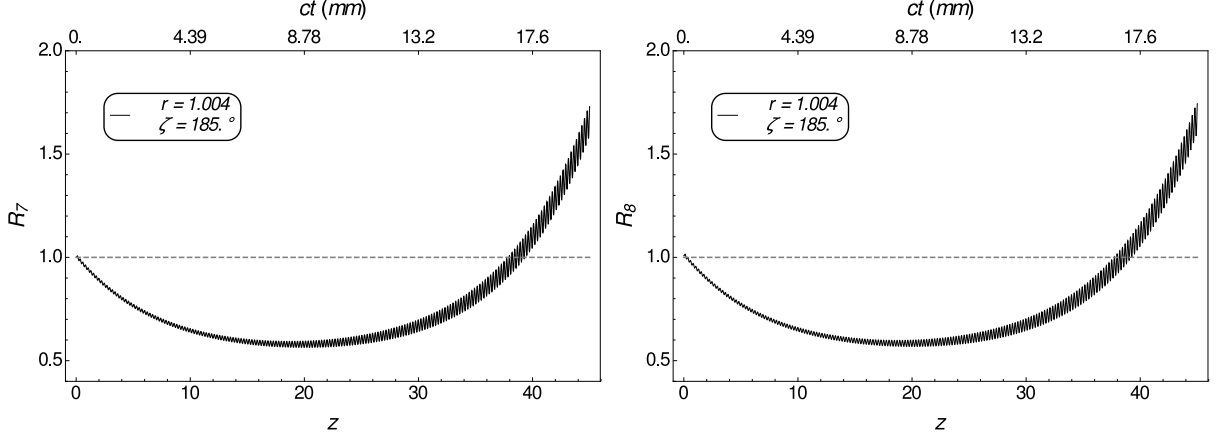


FIG. 7: Functions  $R_{7,8}(x, r, \zeta, \lambda)$  for  $B_s$ -mesons. The scale at the top corresponds to the  $ct$  (mm), while the bottom scale corresponds to time in units of the average lifetime  $z = (\Gamma_H + \Gamma_L) t / 2 = \Gamma t$ , where  $t$  is calculated in the  $B_s$ -meson rest frame. One can see that the time-dependent inequalities (24) are violated (taking the proper  $R_N$  for  $r > 1$  and  $r < 1$ ) in almost all of the experimentally allowed range of  $z$ .

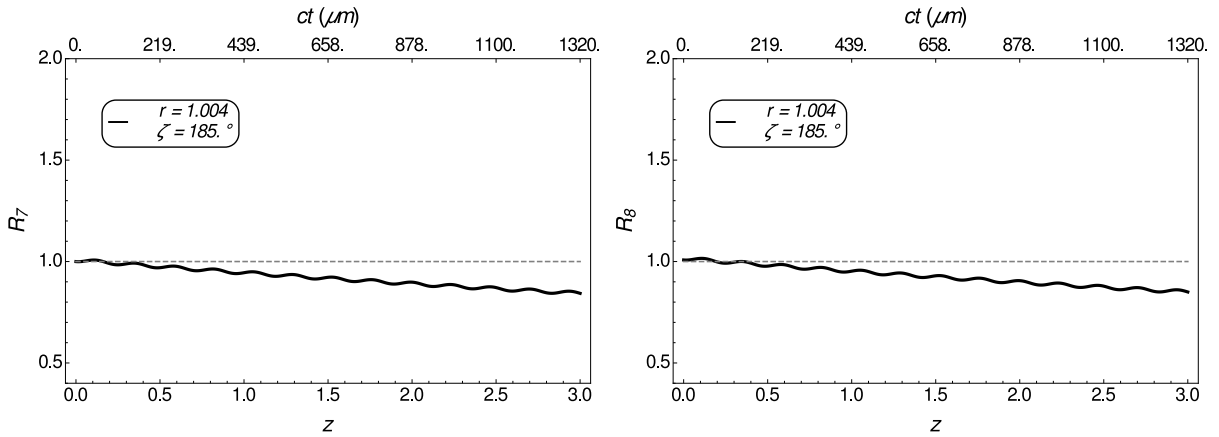


FIG. 8: Functions  $R_{7,8}(x, r, \zeta, \lambda)$  for  $B_s$ -mesons in the range  $z \leq 3$ , which is the most experimentally interesting. Both functions are almost consistent with one, while the  $R_8$  slightly exceeds one at  $z \rightarrow 0$ . Unlike FIG. 7, here the  $ct$  is in microns ( $\mu\text{m}$ ).